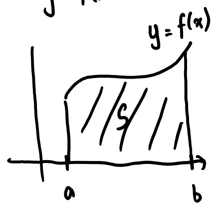
Lecture 22

Tuesday, November 1, 2016 9:27 AM

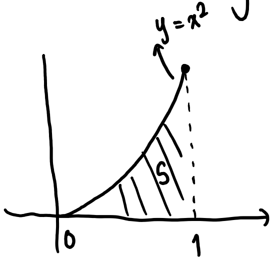
5.1 Areas & Distances

GOAL Find the area of the region S that lies under the curve $y = f(x)$ betn a & b.

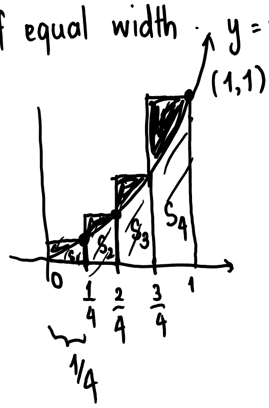


Idea First approximate S by rectangles and take limit of the area of the rectangles as we increase the number of rectangles.

Ex Use rectangles to estimate the area under the curve $y = x^2$ from 0 to 1.



Divide S into four strips S_1, S_2, S_3, S_4 of equal width.



We can approximate each strip by a rectangle whose base is the same as the strip and whose height is the same as the right edge of the strip.

The height of each rectangle are the values of the function $f(x) = x^2$ at the right endpoints of the subintervals $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{2}{4}]$, $[\frac{2}{4}, \frac{3}{4}]$, $[\frac{3}{4}, 1]$.

L

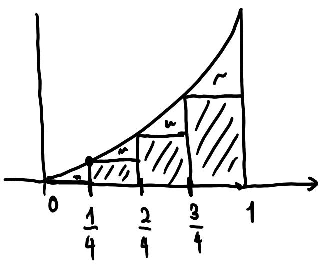
Each rectangle has width $\frac{1}{4}$ and heights $\left(\frac{1}{4}\right)^2$, $\left(\frac{2}{4}\right)^2$, $\left(\frac{3}{4}\right)^2$, $(1)^2$

right endpoints

$$R_4 = \frac{1}{4}\cdot\underbrace{\left(\frac{1}{4}\right)^2} + \frac{1}{4}\cdot\left(\frac{2}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot(1)^2 = 0.46875$$

number of subintervals

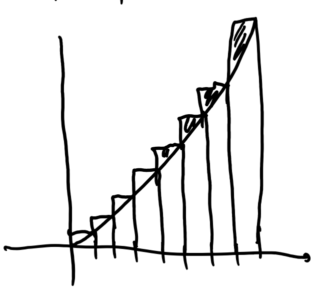
The area A is less that R_4.



$$L_4 = 0^2\cdot\frac{1}{4} + \left(\frac{1}{4}\right)^2\cdot\frac{1}{4} + \left(\frac{2}{4}\right)^2\cdot\frac{1}{4} + \left(\frac{3}{4}\right)^2\cdot\frac{1}{4} = 0.21875$$

$L_4 < A < R_4 \Rightarrow 0.21875 < A < 0.46875$

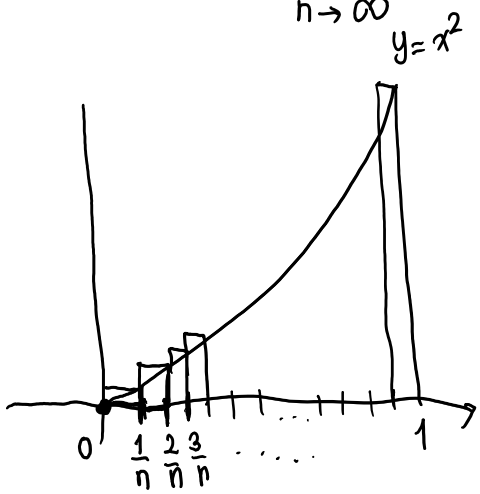
Repeat procedure for 8-subintervals



$R_8 = 0.3984375 > A > L_8 = 0.2734375$

n	L_n	R_n
10	0.285000	0.38500
20	0.30875	0.35875
30	0.3168519	0.3434
100	0.32835	0.33835
1000	0.3328335	0.3338335

GOAL Show $\lim_{n\to\infty} R_n = \frac{1}{3}$



R_n

width $= \frac{1}{n}$

The height of the rectangles are values of function $y = x^2$ at right endpoints $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} = 1$

$$R_n = \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\cdot\frac{1^2}{n^2} + \frac{1}{n}\cdot\frac{2^2}{n^2} + \cdots + \frac{1}{n}\cdot\frac{n^2}{n^2}$$

$$= \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{n^2}{n^3}$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + \cdots + n^2\right]$$

$$= \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{\cancel{n^3}\,n^2}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{1}{n^3} \underline{\sum_{i=1} i^2} = \frac{1}{n^{\cancel{3}2}} \cdot \underline{\frac{\cancel{n}(n+1)(2n+1)}{6}}$$

$$R_n = \frac{2n^2 + 3n + 1}{6n^2}$$

Then $\displaystyle\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{2n^2+3n+1}{6n^2} \overset{L'H}{=}$

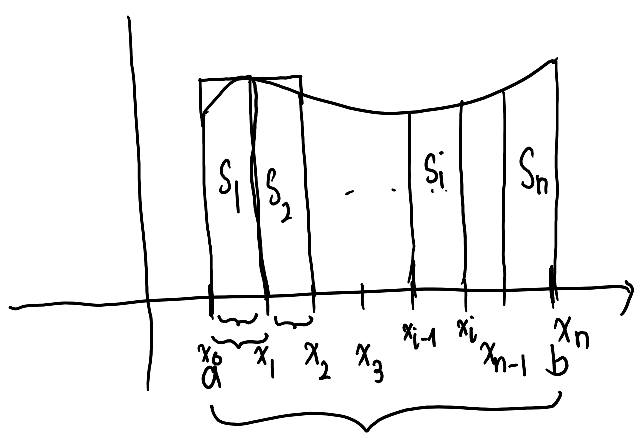
$$= \lim_{n \to \infty} \frac{4n+3}{12n} \overset{L'H}{=}$$

$$= \lim_{n \to \infty} \frac{4}{12} = \frac{1}{3}$$

Similarly we can show $\displaystyle\lim_{n \to \infty} L_n = \frac{1}{3}$

<u>General Case</u>

$y = f(x)$ betn a & b



$$\frac{b-a}{n} = \Delta x$$

<u>1^{st} Strip</u>

Width $= \Delta x$.

Ht $= f(x_1)$.

<u>2^{nd} Strip</u>

Width $= \Delta x$

Ht $= f(x_2)$

<u>i^{th} Strip</u>

Width $= \Delta x$

Ht $= f(x_i)$

$R_n = ?$

Divide $[a,b]$ into n-equal subintervals of width $\Delta x = \frac{b-a}{n}$

$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$

$x_0 = a, \; x_n = b$

$x_1 = x_0 + \Delta x = a + \Delta x$

$x_2 = x_1 + \Delta x = (a + \Delta x) + \Delta x = a + 2\Delta x$

$x_3 = a + 3\Delta x$

$\vdots$

$x_i = a + i \cdot \Delta x$

$$R_n = \Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \Delta x \cdot f(x_3) + \ldots + \Delta x \cdot f(x_n)$$

$$= \Delta x \left[f(x_1) + f(x_2) + \ldots + f(x_n) \right]$$

$$R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} f(x_i)\, \Delta x$$

DEFN The area A of the region S that lies under continuous function f between a and b is

$$A = \lim R_n = \lim \sum^{n} f(x_i)\, \Delta x, \quad \Delta x = \frac{b-a}{},$$

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1} f(x_i) \Delta x , \quad \Delta x = \frac{b-a}{n},$$

$$x_i = a + i \Delta x$$